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EXAMINATION FEBRUARY-MARCH 2024
MASTER OF ARTS EXTERNAL PART-2
MATHEMATICS
ADVANCED FUNCTION ANALYSIS - LEVEL 4

[Time: As Per Schedule] **[Max. Marks:100]**

Instructions: **1. Fill up strictly the following details on your answer book** a. Name of the Examination : **MASTER OF ARTS EXTERNAL PART-2** b. Name of the Subject : **MATHEMATICS ADVANCED FUNCTION ANALYSIS - LEVEL 4** c. Subject Code No : **2101001102040001** 2. Sketch neat and labelled diagram wherever necessary. 3. Figures to the right indicate full marks of the question. 4. All questions are compulsory. 5. Follow usual notations and conventions

Seat No:

Student's Signature

Q.1

(a) Let H be a Hilbert space and M is orthogonal set in H then M is total iff for all non-zero $x \in H$ Parseval's relation $\sum_k |\langle x, e_k\rangle|^2 = \|x\|^2$ holds. **7**

(b) Let H_1 and H_2 be two Hilbert spaces and $T: H_1 \to H_2$ be a bounded linear operator. Then prove that the Hilbert adjoint operator T* of T, an operator from H_2 to H_1 such that $(\langle Tx, y\rangle = \langle x, T^*y\rangle, \forall \in H_1, y \in H_2$, exists, is unique and is a bounded linear operator with norm $\|T^*\| = \|T\|$. **7**

(c) Let X and Y be inner product spaces and Q:X → Y be a bounded linear operator then **6**
(i) $Q = 0 \ iff \ \langle Qx, y\rangle = 0, \forall x \in X, y \in Y$
(ii) $Q: X \to X$ where X is complex and $\langle Qx, x\rangle = 0$ then x=0

OR

(a) Let M be a subset of an inner product space then **7**
I. If M is total in X, then there does not exist a non-zero $x \in X$ which is orthogonal to every element of X. i.e., $x \perp M \Rightarrow x = 0$.
II. If X is complete, the condition is sufficient for totality of M in X.

(b) A bounded linear operator T: X→Y, where X and Y are Banach spaces, is an open mapping. Hence, T is bijective, T^{-1} is continuous and thus bounded. **7**

(c) Prove that the product of two bounded self-adjoint operator S and T on a Hilbert space is self-adjoint iff S and T commutes. **6**

Q.2 (a) Let X be a real vector space and P be a sublinear functional on X. Let f be linear functional defined on a subspace Z of X satisfying $f(x) \leq P(x), \forall x \in Z$ then f has a linear extension $\hat{f}: Z \rightarrow X$ satisfying $\hat{f}(x) \leq P(x), \forall x \in X \text{ and } \hat{f}(x) = f(x), \forall x \in Z.$ **7**

(b) Write Zorn's lemma. Prove that in every Hilbert space $H \neq \{0\}$, there exists total orthonormal set. **7**

(c) Let f be linear functional defined on a subspace Z of a normed space X then there exists a bounded linear functional $\hat{f}$ on X, an extension of f from Z to X with norm $\|\hat{f}\|_X = \|f\|_Z$, where $\|\hat{f}\|_X = Sup\ x \in X_{\|x\|=1}|\hat{f}(x)|$ and $\|f\|_Z Sup\ x \in Z_{\|x\|=1}\ |f(x)|$. **6**

OR

(a) Prove that every Hilbert space H is reflexive. **7**

(b) The adjoint operator $T^{\times}$ of a bounded linear operator T: X→ Y is bounded, linear and $\|T^{\times}\| = \|T\|$, where $T^{\times}: Y' \rightarrow X'$ is defined as $(T^{\times}g)(x) = g(Tx) = f(x), \forall x \in X,, g \in Y' and\ f \in X'$ **7**

(c) Let X be a normed linear space and $x_0\ (\neq 0) \in X$ be arbitrary then there exists a bounded linear functional $\hat{f}$on X such that $\|\hat{f}\| = 1\ and\ \hat{f}(x_0) = \|x_0\|$ **6**

Q.3 (a) Let $T_n \in B(X, Y)$ where X is Banach space and Y is a normed space. If (T_n) is strongly operator convergent with limit T then $T \in B(X, Y)$. **7**

(b) Let T: X→ X be a compact linear operator on a normed space X and $\lambda \neq 0$ then prove that there exists a smallest integer r such that from $n = r$ on, the null spaces $N(T_\lambda^n)$ are all equal, and if $r > 0$, the inclusions $N(T_\lambda^0) \subset N(T_\lambda^1) \subset \cdots N(T_\lambda^r)$ are all proper. **7**

(c) Let X and Y be normed spaces and T: X→ Y a linear operator. Then prove that **6**
(i) If T is bounded and dim T(X) < ∞, the operator T is compact.
(ii) If dim T(X) < ∞, the operator T is compact.

OR

(a) Let Y be a proper closed subspace of a norm space X. Let $x_0 \in X - Y$ be arbitrary and $\delta = inf \|\hat{f} - x_0\|$ be the distance from x_0 to Y then prove that there exists $\hat{f} \in X'$ such that $\|\hat{f}\| = 1, \hat{f}(y) = 0, \forall y \in Y$ and $\hat{f}$ $(x_0) = \delta$. **7**

(b) Let T: X→ Y be a compact linear operator, defined on norm space then prove that its ad joint operator is also compact linear operator. **7**

(c) For every fixed $x \in X$ (where X is a normed space) the functional g_x defined by $g_x(f) = f(x)$ is bounded linear functional on X' and $\|g_x\| = \|x\|$. **6**

Q.4 (a) Prove that every positive bounded self-ad joint linear operator T: H→ H on a complex Hilbert space H has a positive square root A, which is unique. This operator A commutes with every bounded linear operator on H which commutes with T. **7**

(b) Let P_1 and P_2 be projections on a Hilbert space H. Then prove that **7**

(i) The difference $P = P_1 - P_2$ is a projection on H if and only if $Y_1 \subset Y_2$, where $Y_j = P_j(H)$

(ii) If $P = P_2 - P_1$ is a projection, P projects H onto Y, where Y is the orthogonal complement of Y_1 in Y_2.

(c) Let T: X→ X be a compact linear operator from a normed space X. Then for every $\lambda \neq 0$ the null space $\mathcal{N}(T_\lambda) of$ $T_\lambda = T - \lambda I$ is finite dimensional. **6**

OR

(a) The spectrum σ(T) of a bonded self-ad joint linear operator T: H→ H on a complex Hilbert space H is real. **7**

(b) Define Redolent set. Let T: H→ H be a bounded, self-ad joint linear operator on a complex Hilbert space H and $H \neq \{0\}$ then prove that m and M are spectral value of T, where $m = inf_{\|x\|=1}\langle T_x, x\rangle$ and $M = sum_{\|x\|=1}\langle T_x, x\rangle$. **7**

(c) Prove that a bounded linear operator P: H→ H on a Hilbert space H is a projection if and only if P is self-adjoint and idempotent. **6**

Q.5 (a) If the dual space X' of a normed space X is seperable, then X itself is separable. **7**

(b) Let T: X → Y be a compact linear operator from a normed space X into a Banach space Y, then prove that T has compact linear extension. **7**

(c) Let T: X→Y be a compact linear operator from a normed space X into a Banach space Y, then prove that T has compact linear extension. **6**

OR

(a) If a linear operator T is defined on all of a complex Hilbert space H and satisfies $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$, then T is bonded. **7**

(b) Let T: D(T) → H be densely defined linear operator in H and suppose that T is injective and its range R(T) is dense in H then prove that T* is injective and $(T^*)^{-1} = (T^{-1})^*$. **7**

(c) State and prove Weak Convergence theorem. **6**

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